CS3383 Unit 5.0: Backtracking and SAT

David Bremner

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Outline

Combinatorial Search

- Backtracking
- SAT
- Tractable kinds of SAT

N-queens

Problem Description

Given an $n \times n$ chess board, can you place n queens so that no two are in the same row, column, or diagonal.

	Q		
			Q
Q			
		Q	

$\checkmark$

	Q		
			Q
	Q		
		Q	

no

- One per row/col is easy to enforce

Representing Chessboards

- We only care about cases where there is 1 queen per column
- Represent a $n \times n$ board as an array of n integers, meaning which row.
- `None` for not chosen yet.

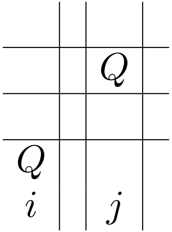
	Q		
			Q
Q			
		Q	

$(1, 3, 0, 2)$

		Q	
Q			

$(0, \texttt{None}, 2, \texttt{None})$

Detecting collisions



$$Q[j] - Q[i] = j - i$$

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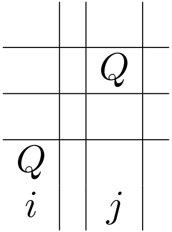

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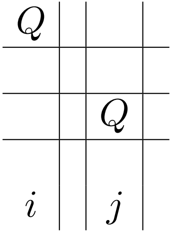

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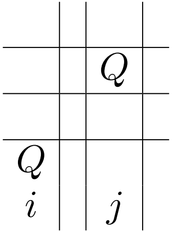

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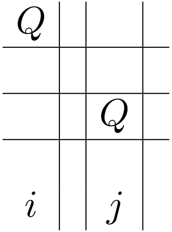

$$Q[j] - Q[i] = i - j$$

- And one more (easy) case

Backtracking Requirements

1. A representation for partial solutions

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Backtracking Requirements

1. A representation for partial solutions
2. A procedure to expand a problem into smaller subproblems
3. A test for partial solutions that returns

`True`	if the solution is complete (Success)
`False`	if there is no way to complete (Failure)
`None`	if neither can be quickly determined. (Uncertainty)

Generic Backtracking

```
def backtrack(P0):
    S = [P0]
    while len(S) > 0:
        P = S.pop()
        result = test(P)
        if result == True:
            return P
        elif result == None:
            for R in expand(P):
                S.append(R)
    return False
```

Backtracking for N-Queens: Framework

representation $Q[1...n]$ where $Q[i]$ is row chosen, or `None`.

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```
def test(Q):
  default = True
  for i in range(len(Q)):
    if Q[i]==None:
      default = None
    else:
      for j in range(i):
        if Q[i] - Q[j] in [0,i-j,j-i]:
          return False
  return default
```

Backtracking for N-Queens: Expand

```
def expand(Q):
  i=0; S=[]
  while Q[i] != None:
    i+=1
  for j in range(len(Q)):
    R=Q[:] # copy
    R[i] = j
    S.append(R)
  return S
```

Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{Z}^+$, T

Decide Is there a subset of X that sums to T

Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{Z}^+$, T

Decide Is there a subset of X that sums to T

- If (X, T) has feasible solution Z, for all $y \in X$, either the solution includes y or not.

Backtracking for SubsetSum

```
def SubsetSum(X,T):
    if T == 0:
        return true
    elif T<0 or len(X) == 0:
        return False

    (y,rest) = (X[0],X[1:])
    return SubsetSum(rest, T-y) \
        or SubsetSum(rest,T)
```

The SAT Problem

Conjunctive Normal Form (CNF)

Variables $\{ x_1 ... x_n \}$

Literals $L = \{ x_i, \bar{x}_i \mid \text{variable } x_i \}$

Clause $\{ z_1, ..., z_k \} \subset L$

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Propositional Satisfiability (SAT)

Instance Set of clauses S

Question $\exists$ setting of variables to 0, 1 such that each clause has at least one true literal?

SAT Example

$$\{\,\{\,1,2,3\,\},\{\,-1,-2,-3\,\}\,\} = \{\,\{\,x_1,x_2,x_3\,\},\{\,\bar{x}_1,\bar{x}_2,\bar{x}_3\,\}\,\}$$

(A) $=$

Truth Table

x_1	x_2	x_3	A
0	0	0	0
0	0	1	1
0	1	0	
	$\vdots$		

SAT Example

$$\{\,\{\,1, 2, 3\,\}, \{\,-1, -2, -3\,\}\,\} = \{\,\{\,x_1, x_2, x_3\,\}, \{\,\bar{x}_1, \bar{x}_2, \bar{x}_3\,\}\,\}$$

$$\text{(A)} \qquad = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

Truth Table

x_1	x_2	x_3	A
0	0	0	0
0	0	1	1
0	1	0	
	$\vdots$		

Backtracking for SAT

representation	(reduced) clauses
test	if empty clause, return `False`. If no clauses, return `True`. Otherwise return `None` (UNKNOWN)
expand	$P_0 = \text{reduce}(P, x_j)$, $P_1 = \text{reduce}(P, \bar{x}_j)$ for some j.

Backtracking SAT Example II

$$(w \vee x \vee y \vee z), (w \vee \overline{x}), (x \vee \overline{y}), (y \vee \overline{z}), (z \vee \overline{w}), (\overline{w} \vee \overline{z})$$

$$w = 0$$

$$(x \vee y \vee z), (\overline{x}), (x \vee \overline{y}), (y \vee \overline{z})$$

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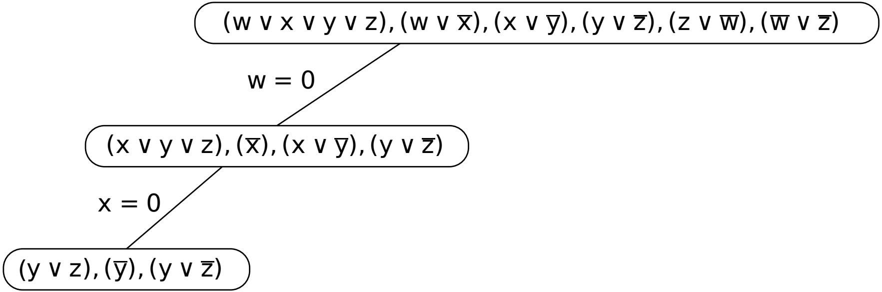

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$y = 0$

$(z), (\overline{z})$

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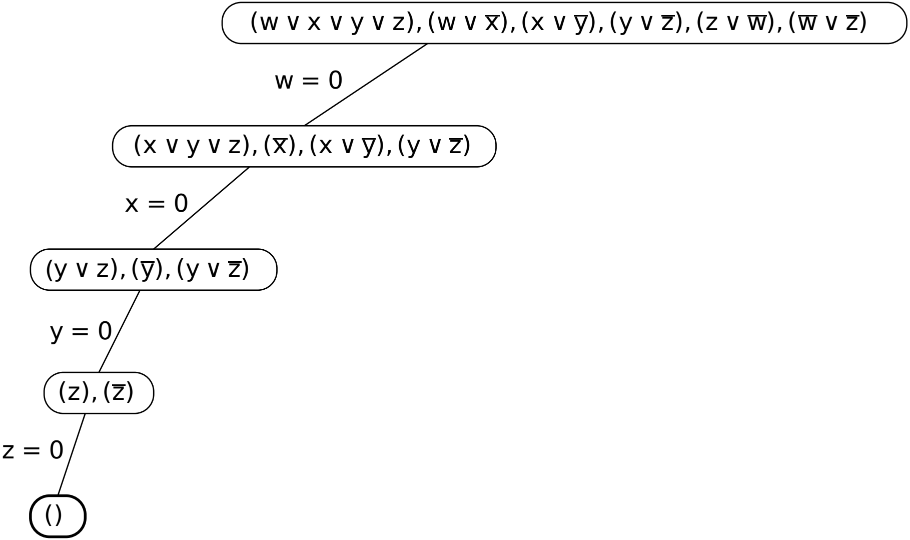

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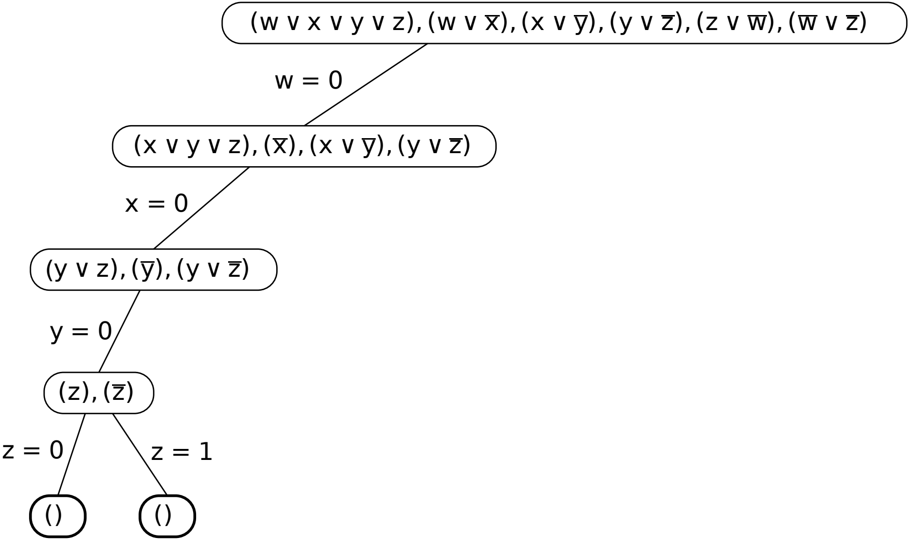

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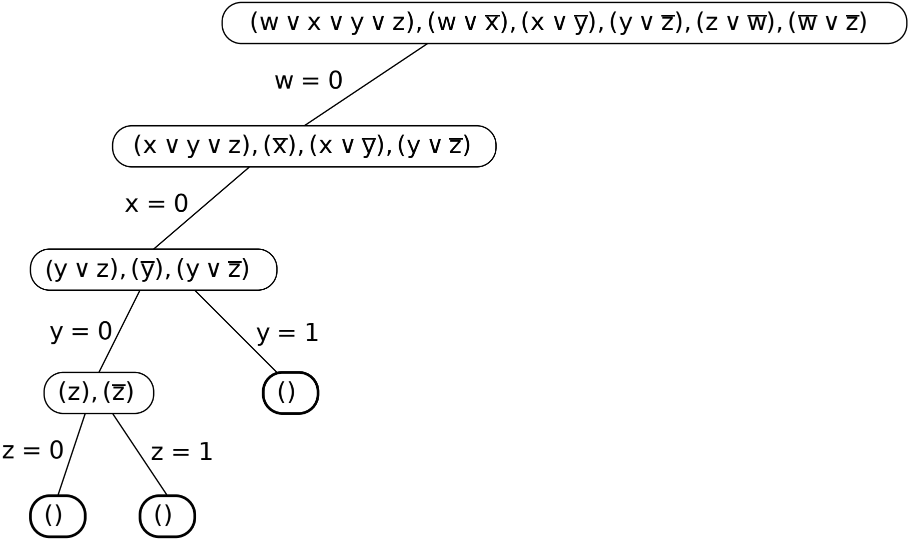

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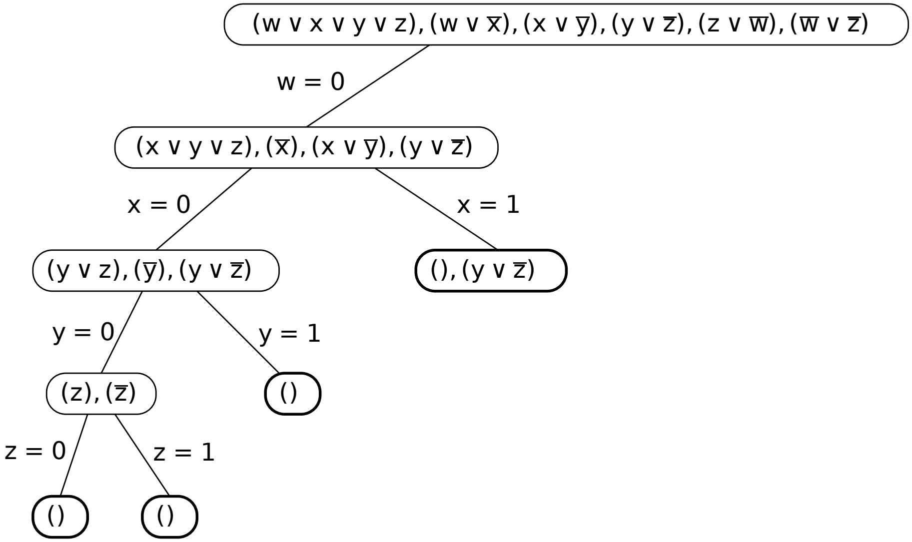

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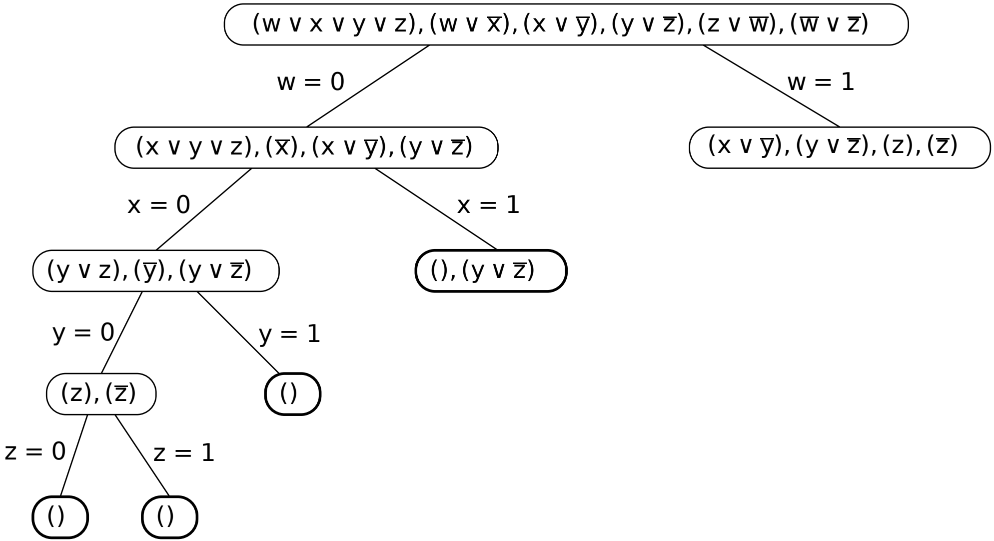

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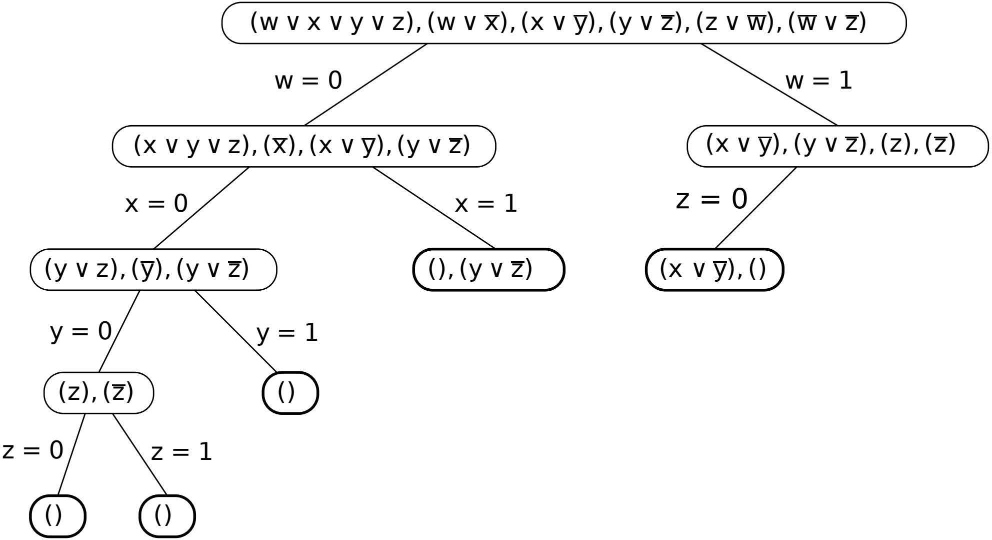

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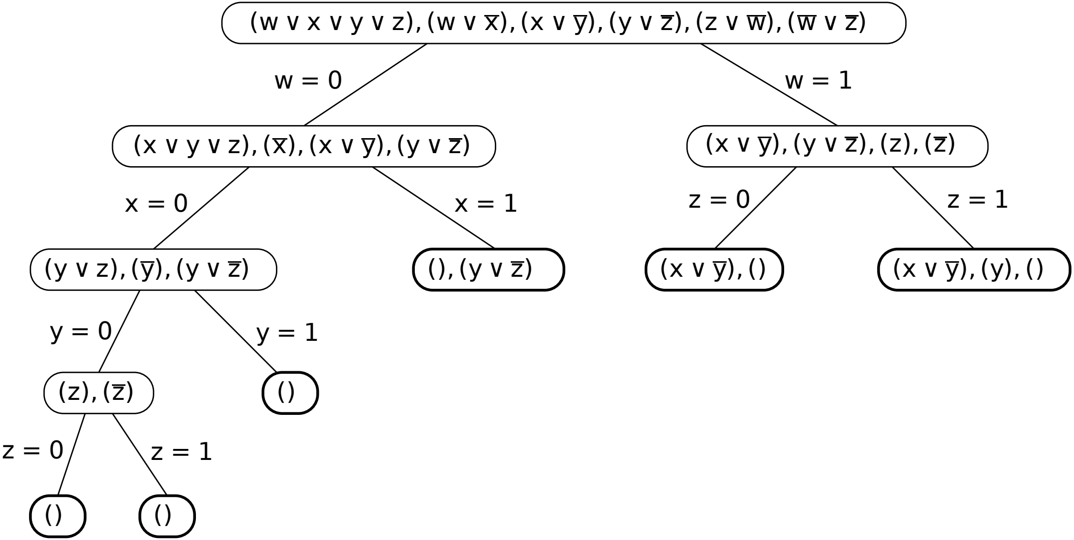

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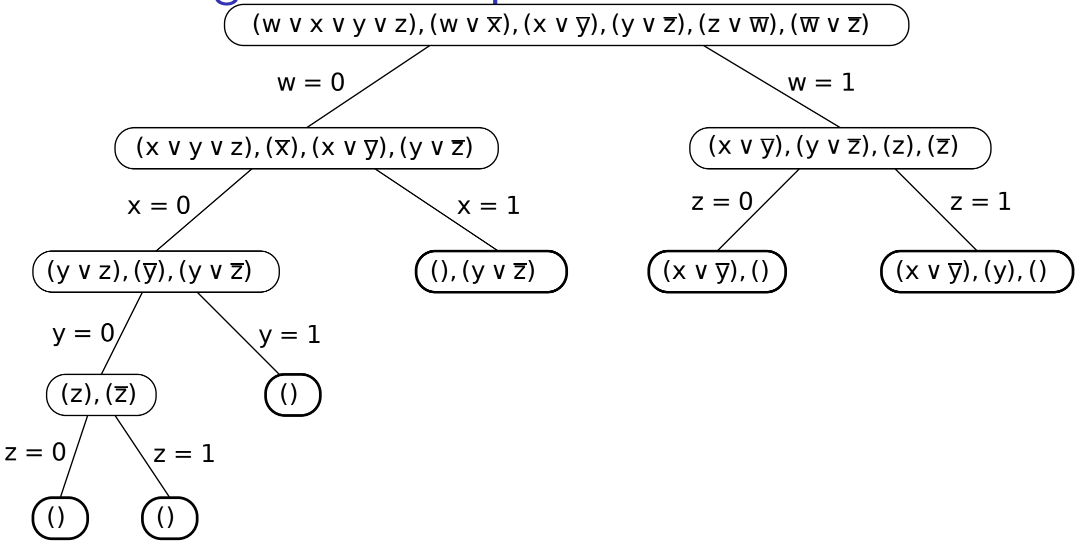

We tried 11 possibilities. Maximum?

Backtracking Sat test

```python
def test(clauses):
    if (len(clauses)) == 0:
        return True
    for clause in clauses:
        if len(clause)==0:
            return False
    return None
```

Backtracking Sat expand

```
def reduce(clauses,literal):
  out=[]
  for C in clauses:
    if not literal in C:
      new=[z for z in C if z != -1*literal]
      out.append(new)
  return out

def expand(clauses):
  j = clauses[0][0]
  return [reduce(clauses,j),
          reduce(clauses,-j)]
```

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- to maximize number of clauses satisfied, choose $x_1 \leftarrow 1$, $x_4 \leftarrow 0$
- solvable with *unit propagation*

Horn SAT

Horn formulas

implication $(z \wedge w \wedge q) \Rightarrow u$. LHS is all positive, RHS one positive literal

negative clauses $(\bar{x} \vee \bar{w} \vee \bar{y})$. All literals negated.

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$$\left(\bigwedge_{i=1}^{k} x_i\right) \Rightarrow y$$

$$\neg\left(\bigwedge_{i=1}^{k} x_i\right) \vee y$$

$$\left(\bigvee_{i=1}^{k} \bar{x}_i\right) \vee y$$

Horn formulas as CNF

- negative clauses are already CNF
- implications use the following transformations

$$(\bigwedge_{i=1}^{k} x_i) \Rightarrow y$$
$$\neg(\bigwedge_{i=1}^{k} x_i) \vee y$$
$$(\bigvee_{i=1}^{k} \bar{x}_i) \vee y$$

- special CNF with at most one positive literal.

Unit propagation

```
def UnitProp(S):
  Q = [ c for c in S if len(c)==1 ]
  while len(Q)>0:
    z = Q.pop()[0]; T = []
    for C in S:
      C = [j for j in C if j!=-z]
      if len(C)==0: return False
      if len(C)==1: Q.append(C)
      if not z in C: T.append(C)
    if len(T)==0: return True
    S = T
  return S
```

Unit propagation (with assignment)

```
def UnitProp(S):
  Q = [ c for c in S if len(c)==1 ]; V = []
  while len(Q)>0:
    z = Q.pop()[0]; T = []
    V.append(z)
    for C in S:
      C = [j for j in C if j!=-z]
      if len(C)==0: return (False,V)
      if len(C)==1: Q.append(C)
      if not z in C: T.append(C)
    if len(T)==0: return (True,V)
    S = T
  return S
```

Solving Horn SAT with Unit Propagation

1. Apply unit propagation
2. If no contradiction is detected, set the remaining variables to false.

Claim 1

If the procedure detects a contradiction, the instance is unsatisfiable

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Claim 2

If no contradiction is detected, the resulting assignment is valid

Proof

any remaining clause has at least one negative literal

Solving 2SAT with Unit Propagation

Claim

Applying `reduce`, followed by unit propagation, always yields either a contradiction (empty clause), or a subset of the original clauses.

Solving 2SAT with Unit Propagation

```
def two_sat(clauses):
  if len(clauses) == 0:
      return True
  j = clauses[0][0]
  R0 = UnitProp(reduce(clauses,-j))
  R1 = UnitProp(reduce(clauses,j))
  if True in [ R0, R1 ]: return True
  if R0 == False and R1 == False: return False
  if R0 == False:
      return two_sat(R1)
  else:
      return two_sat(R0)
```

2SAT with Unit Propagation

Correctness

- if `two_sat` returns `False`, the formula is unsatisfiable.
- if `two_sat` returns `True`, the formula is satisfiable.

By induction on number of clauses; base case: no clauses. Suppose the function is correct for $j < k$ clauses.

return `False`

directly both choices for x_j led to a contradiction.

indirectly we found a contradiction in some subset of the original clauses.

2SAT with Unit Propagation

Correctness

- if `two_sat` returns `False`, the formula is unsatisfiable.
- if `two_sat` returns `True`, the formula is satisfiable.

By induction on number of clauses; base case: no clauses. Suppose the function is correct for $j < k$ clauses.

return `True`

directly
: one of our choices for x_j, along with unit prop., satisfied all clauses.

indirectly
: all clauses containing x_j removed. remaining clauses satisfiable by induction.

Backtracking Sat with Unit Propagation

```python
def backtrack(P0):
    S = [P0]
    while len(S) > 0:
        P = S.pop()
        P1 = UnitProp(P)
        if P1 != False:
            if P1 == True:
                return P1
            else:
                for R in expand(P1):
                    S.append(R)
    return False
```