CS3383 Unit 2.4: Union Find Path Compression

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February 26, 2024

Outline

Union Find
- Path Compression
- Path Compression Analysis

"Memoizing" the find routine

```
def find(P, key):
  while P.parent[key] != key:
    key = P.parent[key]
  return key
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def find(P, key):
  if P.parent[key] != key:
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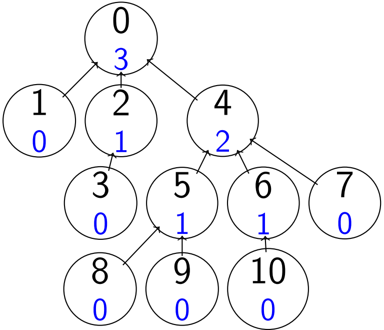

before

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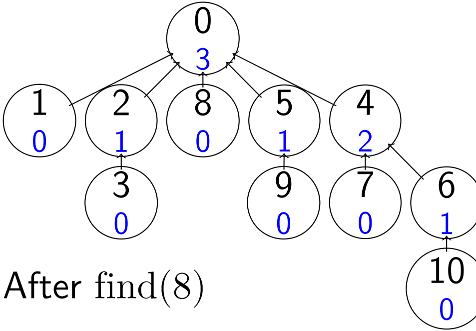
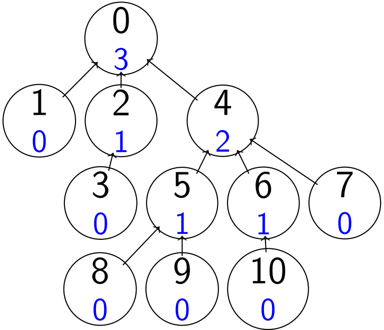
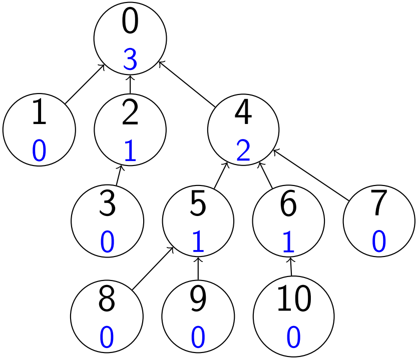


After find(8)



before

Find example



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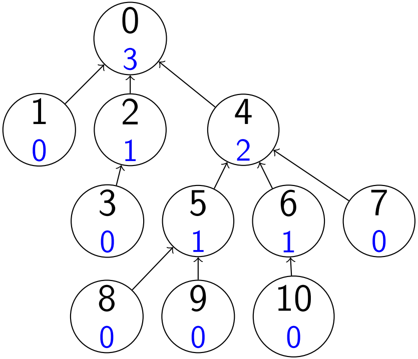
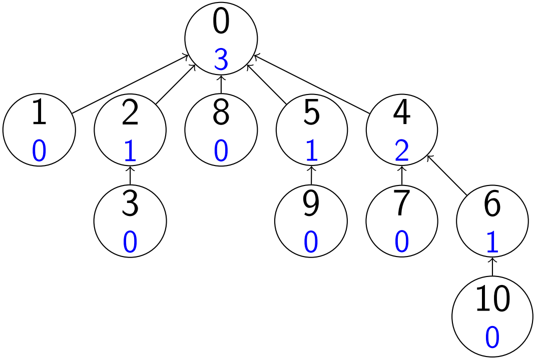
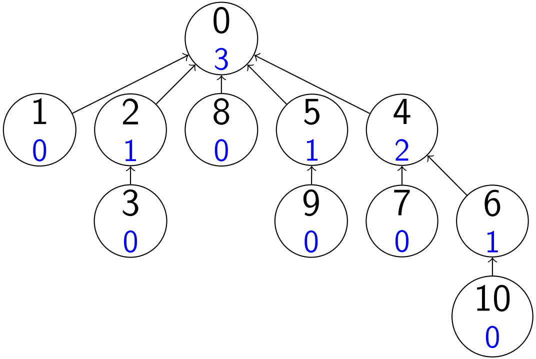




After find(8)

find(8), find(10)



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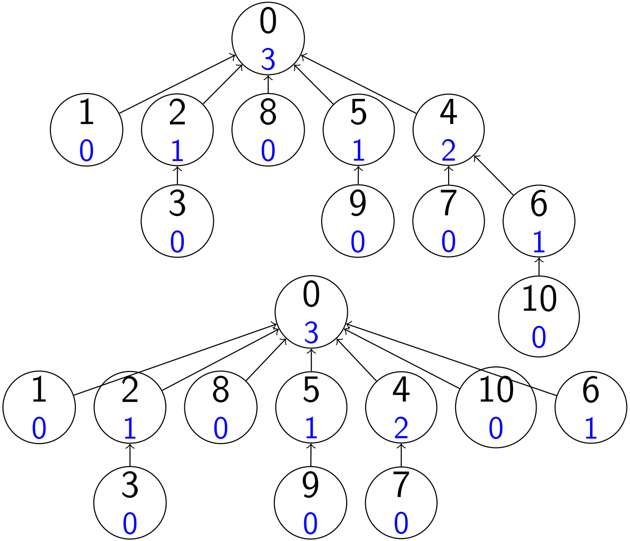

Rank ordering is maintained

Property 1

For any x such that $\mathrm{parent}(x) \neq x$,
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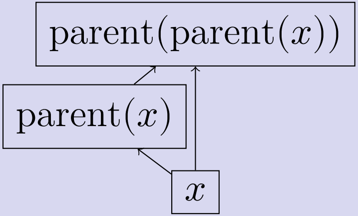
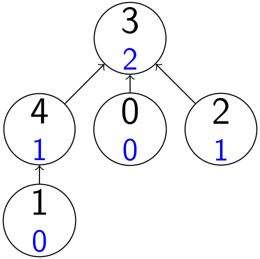
Shortcuts preserve order



Size of trees is preserved, but not subtrees.

Property 2'

Any root node of rank k has at least 2^k nodes in its subtree.



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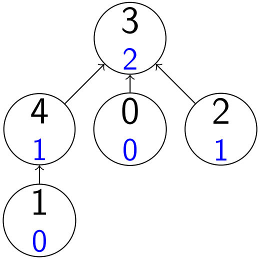
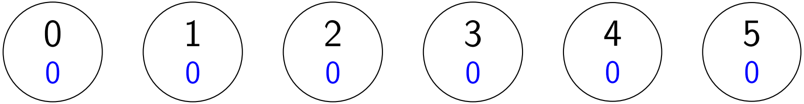
Proof of property 2'.

Induction: Base case is $k = 0$. Roots of rank k are made from two rank $k - 1$ roots. □



Union+Find Example 1/

- initial partition



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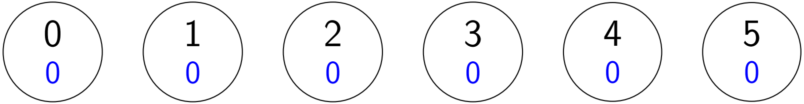
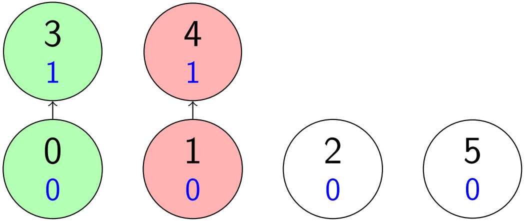
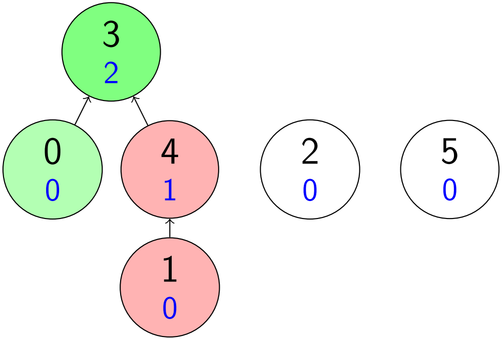


- after union(0,3), union(1,4)



Union+Find Example 2/

after union(4,0)



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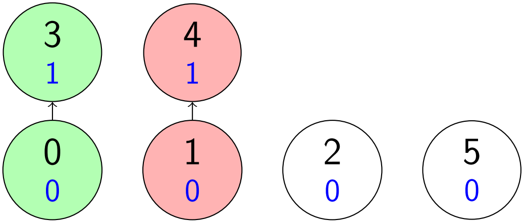
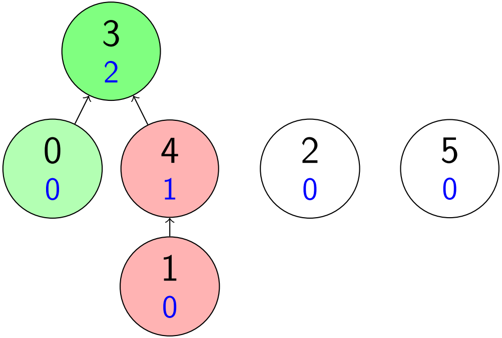
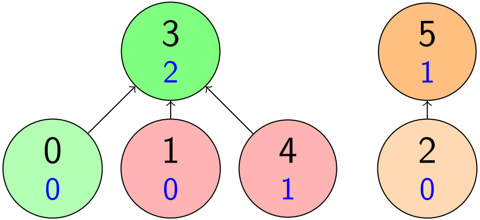


Union+Find Example 3/

after union(4,0)



after union(4,0), find(1), union(2,5)



Union+Find Example 4/

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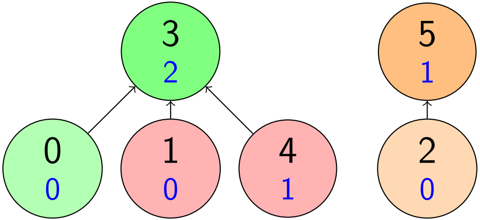
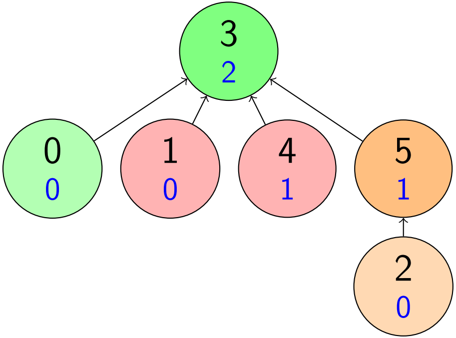
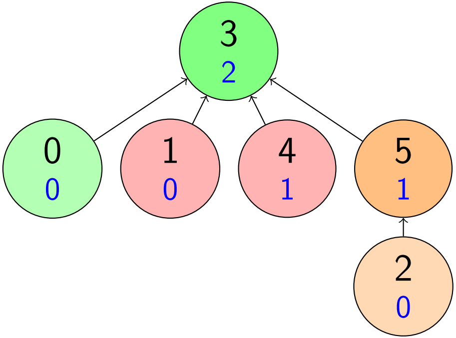


after union(5,0)

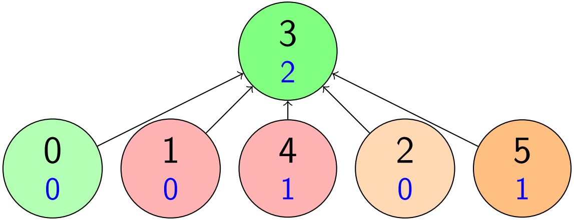


Union+Find Example 5/

after union(5,0)



after union(5,0), find(2)



Not too many nodes of rank k

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- From property 1, descendents of a given rank k node are distinct.
- When a node gets rank $k > 0$, it is a root, and has 2^k descendents.
- Those descendents are never used to make another node rank k. (non-roots stay non-roots).

Rank intervals

- We divide the numbers $[1, n]$ into $[k+1, 2^k]$

$$[1,1],[2,2],[3,4],[5,16],\ldots,[k+1,2^k]$$

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- $\log^*(n)+1$ intervals cover n

$$\log^*(n) = \begin{cases} 1 & \text{if } \log(n) \leq 1 \\ 1+\log^*(\log(n)) & \text{otherwise} \end{cases}$$

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$$= \frac{1}{2^{k+1}} \frac{1}{1-1/2} \qquad \text{G.S.}$$

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Paying for find operations 1/2

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- If in the same interval, we say key pays a dollar back.

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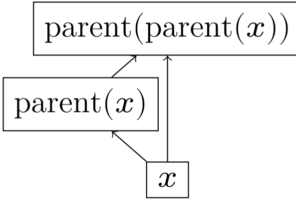

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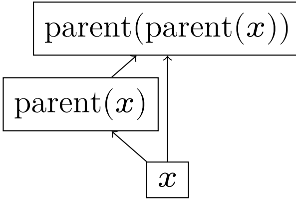

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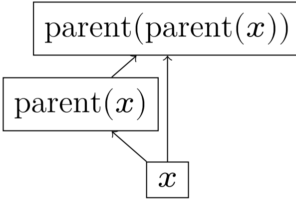

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- If $\mathrm{rank}(x) \in [k+1 \ldots 2^k]$, that can repeat less than 2^k times before its parent is in a higher interval.
- Once that happens, payments stop.

Summing up

- We can think about the analysis as classifying all of the updates to a given key as "near" or "far", and bounding those in two different ways.
- Total cost for n operations
  - $\leq n \log^* n$ total steps where parent is in next interval
  - $\leq n \log^* n$ total steps where parent is in same interval
- Amortized cost in $O(\log^* n)$ per operation.