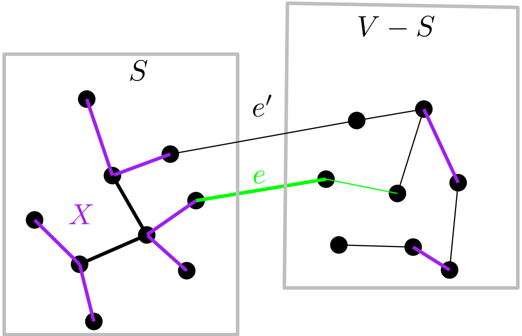
Outline

Cut Property

Lemma

Let T be a minimum spanning tree, $X \subset T$ s.t. X does not connect $(S, V - S)$. Let e be the lightest edge from S to $V - S$. $X \cup e$ is part of some MST.

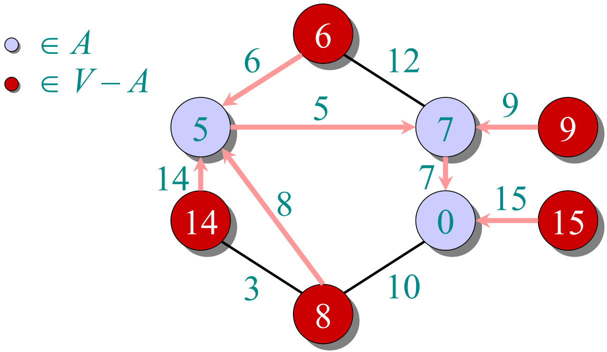


Prim's Algorithm

```
def prim(G,root):
  pq = pqdict(); prev = {}
  for v in G.keys():
    pq.additem(v,inf)
  pq.updateitem(root,0)
  while len(pq)>0:
    v = pq.pop()
    for (z,weight) in G[v]:
      if z in pq and weight < pq[z]:
        prev[z]=v
        pq.updateitem(z,weight)
  return prev
```

Cut Property and Prims Algorithm

$$S := V - \texttt{pq}$$
$$X := \{(u, v) \in S \times S \mid v = \texttt{prev}[u]\}.$$



Cut Property and Prims Algorithm

$$S := V - \mathtt{pq}$$
$$X := \{(u, v) \in S \times S \mid v = \mathtt{prev}[u]\}.$$

Loop Invariant / Inductive Hypothesis

1. X is a subset of some MST
2. For z non-root, pq$[z]$ is weight of the cheapest crossing edge ending at z.

Cut Property and Prim's Algorithm

```
def prim(G,root):
  ⋮
  while len(pq)>0:
    # S ← S ∪ {v}, X ← X ∪ {(prev[v], v)}
    v = pq.pop()
    for (z,weight) in G[v]:
      if z in pq and weight < pq[z]:
        # found a cheaper crossing edge to z
        prev[z]=v
        pq.updateitem(z,weight)
  return prev
```

Prim's induction

L.I. (1) $X \subseteq$ MST. (2) $\text{pq}[z] =$ cheapest ce to z.

Base Case

$X = \emptyset$

Prim's induction

L.I. (1) $X \subseteq$ MST. (2) $\text{pq}[z] =$ cheapest ce to z.

Induction

Suppose after $k \geq 0$ iterations, L.I. holds. Iteration $k+1$:

Prim's induction

L.I. (1) $X \subseteq$ MST. (2) $\mathtt{pq}[z]$ = cheapest ce to z.

Induction

Suppose after $k \geq 0$ iterations, L.I. holds. Iteration $k+1$:

LI1 From L.I.2, we add the cheapest x-ing edge $e = (\mathtt{prev}[v], v)$ to X. By C.P. $X \cup \{e\}$ is part of MST

Prim's induction

L.I. (1) $X \subseteq$ MST. (2) $\mathrm{pq}[z] =$ cheapest ce to z.

Induction

Suppose after $k \geq 0$ iterations, L.I. holds. Iteration $k+1$:

- LI1 From L.I.2, we add the cheapest x-ing edge $e = (\mathtt{prev}[v], v)$ to X. By C.P. $X \cup \{e\}$ is part of MST
- LI2 Only crossing edges starting at v are new in this iteration, and those are updated correctly.