

CS3383 Lecture 1.2: Substitution method

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Outline

Even More Divide and Conquer

- Substitution Method for recurrences

- Substitution examples

Example recurrence

- ▶ The Master Method actually works for this, but it won't always.

$$\begin{aligned}T(n) &= 4T(n/2) + n \\T(1) &= 1\end{aligned}$$

- ▶ Suppose that we want to prove $T(n) \in O(n^3)$ by **induction**
- ▶ Guess $T(n) \leq cn^3$



Example of substitution

$$\begin{aligned}T(n) &= 4T(n/2) + n \\&\leq 4c(n/2)^3 + n \\&= (c/2)n^3 + n \\&= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\&\leq cn^3 \leftarrow \textit{desired}\end{aligned}$$

whenever $(c/2)n^3 - n \geq 0$, for example,
if $c \geq 2$ and $n \geq 1$.

residual



Example (continued)

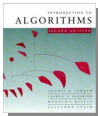
- We must also handle the initial conditions, that is, ground the induction with base cases.
- **Base:** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \leq n < n_0$, we have “ $\Theta(1)$ ” $\leq cn^3$, if we pick c big enough.



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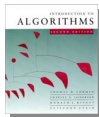
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This bound is not tight!



A tighter upper bound?

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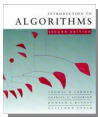
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~~$= O(n^2)$~~ **Wrong!** We must prove the I.H.

$$= cn^2 - (-n) \quad [\text{desired} - \text{residual}]$$

$$\leq cn^2 \quad \text{for no choice of } c > 0. \text{ Lose!}$$



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IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \leq c_1 k^2 - c_2 k$ for $k < n$.



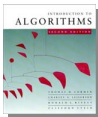
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$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &= 4(c_1(n/2)^2 - c_2(n/2)) + n \\ &= c_1 n^2 - 2c_2 n + n \\ &= c_1 n^2 - c_2 n - (c_2 n - n) \\ &\leq c_1 n^2 - c_2 n \quad \text{if } c_2 \geq 1. \end{aligned}$$



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Pick c_1 big enough to handle the initial conditions.

Substitution example II

$$T(0) = 1$$

$$T(n) = T(n-1) + c^n \quad n > 0, c > 1$$

$$= T(n-2) + c^{n-1} + c^n$$

$$= \sum_{i=0}^n c^i \quad \text{guess!}$$

$$= \frac{c^{n+1} - 1}{c - 1} \quad \text{geo. series}$$

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base $T(0)$

induction apply recurrence to prove GS formula

Substitution example III

$$T(n) = T(n/5) + T(3n/4) + cn$$

$$T(n) \leq dn, n \geq n_0$$

$$\leq (1/5)dn + (3/4)dn + cn$$

$$\leq dn$$

(Guess)

(Strong induction)

(Choice of c, d)