CS3383 Lecture 1.1: The Master Theorem with applications

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Outline

Divide and Conquer Continued

- The Master Theorem
- Matrix Multiplication

The Master Theorem

If $\exists$ constants $b > 0$, $s > 1$ and $d \geq 0$ such that $T(n) = b \cdot T(\lceil \frac{n}{s} \rceil) + \Theta(n^d)$, then

(Simplified from Theorem 4.1 in CLRS4)

The Master Theorem

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$$T(n) = \begin{cases} \Theta(n^d) & \text{if } d > log_s b \quad (\text{equiv. to } b < s^d) \\ \Theta(n^d \log n) & \text{if } d = log_s b \quad (\text{equiv. to } b = s^d) \\ \Theta(n^{\log_s b}) & \text{if } d < log_s b \quad (\text{equiv. to } b > s^d) \end{cases}$$

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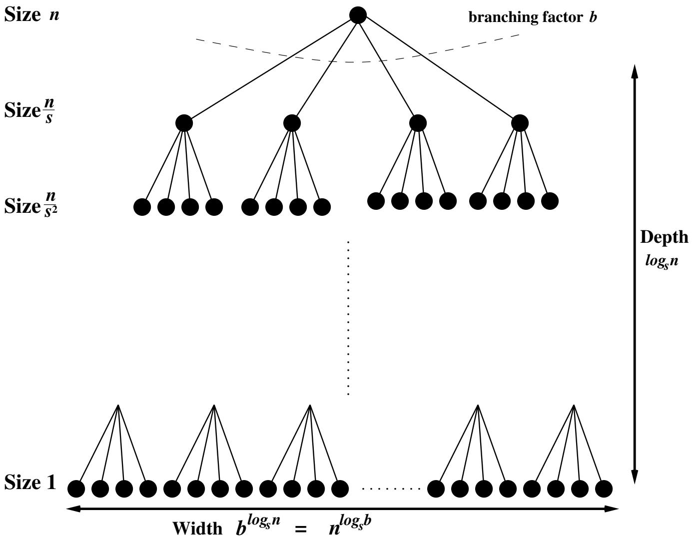
Master theorem, in pictures



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And so

$$T(n) = \sum_{i=0}^{\log_s n} c \cdot \left(\frac{n}{s^i}\right)^d \cdot b^i$$

Proof of Master theorem, $b = s^d$

$$T(n) = \sum_{i=0}^{\log_s n} c \cdot \left(\frac{n^d}{(s^i)^d}\right) \cdot b^i \; = \; c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} \left(\frac{b}{s^d}\right)^i\right)$$

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If $b = s^d$, then

$$T(n) = c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} 1\right) \;=\; c \cdot n^d \log_s n$$

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so $T(n)$ is $\Theta(n^d \log n)$.

Proof of Master Theorem $b \neq s^d$ (1 of 2)

Otherwise ($b \neq s^d$), we have a geometric series,

$$T(n) = c \cdot n^d \cdot \left(\frac{\left(\frac{b}{s^d}\right)^{\log_s n+1} - 1}{\frac{b}{s^d} - 1} \right)$$

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$$T(n) = \frac{s^d}{b - s^d} \cdot c \cdot n^d \cdot \left(\left(\frac{b}{s^d}\right)^{\log_s n+1} - 1 \right)$$

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$$T(n) = \frac{s^d}{b - s^d} \cdot c \cdot n^d \cdot \left(\left(\frac{b}{s^d}\right)^{\log_s n+1} - 1 \right)$$

$$= \frac{s^d}{b - s^d} \cdot c \cdot n^d \cdot \left(\frac{b}{s^d}\right)^{\log_s n+1} - \frac{s^d}{b - s^d} \cdot c \cdot n^d$$

Proof of Master Theorem $b \neq s^d$ (2 of 2)

From rules of powers and logarithms:

$$\left(\frac{b}{s^d}\right)^{\log_s n+1} = \frac{b}{s^d} \cdot \left(\frac{b}{s^d}\right)^{\log_s n} = \frac{b}{s^d} \cdot \frac{b^{\log_s n}}{\left(s^d\right)^{\log_s n}}$$

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$$= \frac{b}{b - s^d} \cdot c \cdot n^{\log_s b} - \frac{s^d}{b - s^d} \cdot c \cdot n^d$$

Branching versus subproblem size 1/2

$$T(n) = \frac{b}{b - s^d} \cdot c \cdot n^{\log_s b} - \frac{s^d}{b - s^d} \cdot c \cdot n^d$$

Now we need to test b versus s^d.

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If $b > s^d$ ($\log_s b > d$), first term dominates:

$$\begin{aligned} T(n) &= c_2 n^{\log_s b} - c_3 n^d && (c_2 > c_3 > 0) \\ &\leq c_2 n^{\log_s b} && (\mathrm{O}) \\ &\geq (c_2 - c_3) n^{\log_s b} && (\Omega) \end{aligned}$$

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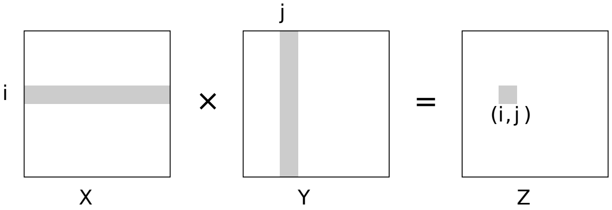
new first term dominates, same argument: $\Theta(n^d)$.

Matrix Multiplication

The product of two $n \times n$ matrices X and Y is a third $n \times n$ matrix $Z = XY$, with

$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}$$

where Z_{ij} is the entry in row i and column j of matrix Z.



Calculating Z directly using this formula takes $\Theta(n^3)$ time.

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Decompose the input matrices into four blocks each

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$$= \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

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8 subinstances AE, BG, AF, BH, CE, DG, CF, DH

Matrix Multiplication: Blocks

8 subinstances of dimension $\frac{n}{2}$, and taking cn^2 time to add the results:

$$T(n) \;=\; 8 \cdot T\left(\frac{n}{2}\right) + cn^2$$

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As with integer mult., naive split does not improve running time.

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where

$$\begin{aligned} P_1 &= A(F - H) & P_5 &= (A + D)(E + H) \\ P_2 &= (A + B)H & P_6 &= (B - D)(G + H) \\ P_3 &= (C + D)E & P_7 &= (A - C)(E + F) \\ P_4 &= D(G - E) \end{aligned}$$

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input size is $m = n^2$, time is $\Theta(m^{1.404})$ time (vs $\Theta(m^{1.5})$).

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