CS3383 Unit 0: Deeper into Asymptotics

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Outline

New asymptotic classes

- Big-theta
- little-o
- little-omega

Exponentials and logs

- Two examples
- The limit of ratios approach

big-Θ example

$$\frac{n^2}{2} - 2n \in \Theta(n^2)$$

- O clear
- Ω: common factors, choose c as an arbitrary constant at most $1/2$

little-o Example 1

$f(n) \in o(g(n))$

$$\forall c > 0 \; \exists n_0 \quad \forall n > n_0 \quad 0 \leq f(n) < cg(n)$$

$$2n^2 \in o(n^3)$$

- find n_0 for arbitrary (unknown) c.

little-o Example 2

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big-O	solve for c
little-o	lower-bound c

little-ω and big-Ω

$f(n) \in \Omega(g(n))$

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little-ω implies big-Ω

Definition ($f(n) \in \omega(g(n))$)

$$\forall c > 0 \; \exists n_0 \quad \forall n > n_0 \quad 0 \le cg(n) < f(n)$$

- assume little-ω, prove big-Ω

little-ω example

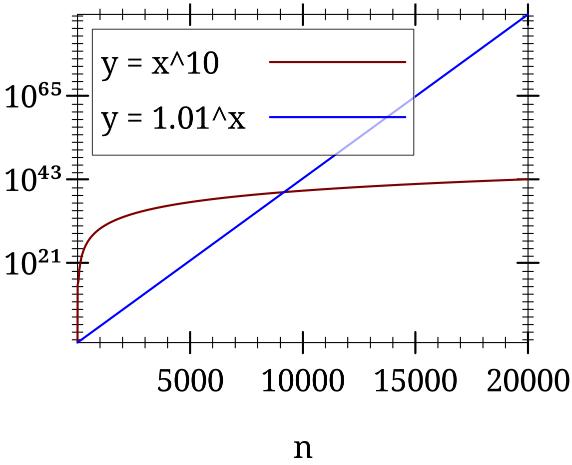
$f(n) \in \omega(g(n))$

$$\forall c > 0 \; \exists n_0 \quad \forall n > n_0 \quad 0 \leq cg(n) < f(n)$$

$$n^3 \in \omega(\frac{n^2}{2} + 2n)$$

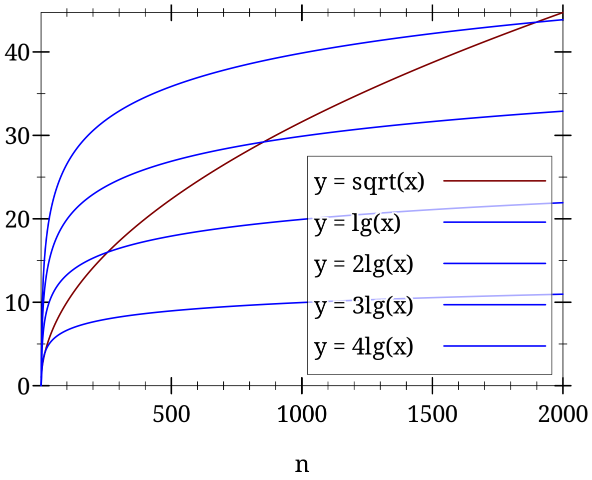
- find n^* s.t. $n^2/2 > 2n$, $\forall n \geq n^*$
- find $n_0 \geq n^*$ s.t. $n^3 > cn^2$, $\forall n \geq n_0$

Exponential versus Polynomial



$$(1.01)^n \in \omega(n^{10}) \subseteq \Omega(n^{10})$$

Root vs log



$$\sqrt{n} \in \omega(\lg n) \subseteq \Omega(\lg n)$$

little-ω redux

$f(n) \in \omega(g(n))$

$$\forall c > 0 \; \exists n_0 \forall n > n_0 \quad cg(n) < f(n)$$

equivalently $\displaystyle \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

(Assume positive functions)

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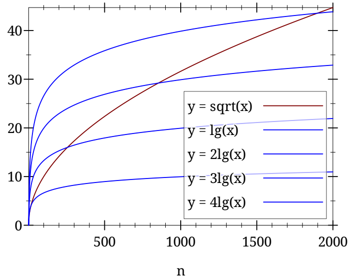
idea of equivalence

$$f(n) > cg(n) \Leftrightarrow \frac{f(n)}{g(n)} > c$$

(Assume positive functions)



Limit of ratios rule

From CLRS (3.13) for $a > 1$:

$$\lim_{n \to \infty} \frac{a^n}{n^b} = \infty$$

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i.e.

$$(1.01)^n \in \omega(n^{10})$$

Proving the limit of ratios rule

$$\lim_{n\to\infty} \frac{a^n}{n^b} =$$

$$=$$

$$=$$

L'Hôpital's rule

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Proving the limit of ratios rule

$$\lim_{n\to\infty} \frac{a^n}{n^b} = \lim_{n\to\infty} \frac{a^n \ln a}{bn^{b-1}}$$

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$$= \lim_{n\to\infty} \frac{a^n \ln^2 a}{b(b-1)n^{b-2}}$$
$$= \lim_{n\to\infty} \frac{a^n \ln^b a}{b!}$$

L'Hôpital's rule

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Another limit ratio rule

$$\lim_{m\to\infty} \frac{m^c}{(\lg m)^b} =$$

$$=$$

$$= \infty \qquad \text{(apply lrr)}$$

Another limit ratio rule

$$\lim_{m\to\infty} \frac{m^c}{(\lg m)^b} = \lim_{m\to\infty} \frac{(2^c)^{\lg m}}{(\lg m)^b}$$

$$=$$

$$= \infty \qquad \text{(apply lrr)}$$

$$m = 2^{\lg m}$$

$$n :=$$

$$a :=$$

$$c > 0 \implies$$

$$m \to \infty \implies$$

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$$= \infty \qquad \text{(apply lrr)}$$

$$m = 2^{\lg m}$$

$$n := \lg m$$

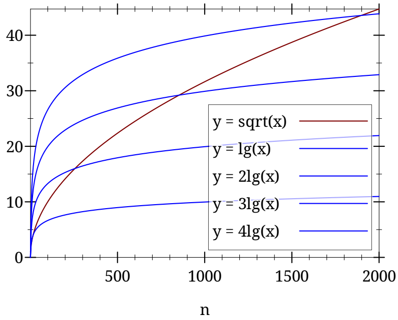
$$a := 2^c$$

$$c > 0 \implies a = 2^c > 1$$

$$m \to \infty \implies n = \lg m \to \infty$$

Log vs Root revealed

$$\sqrt{n} \in \omega(\lg n) \subseteq \Omega(\lg n)$$



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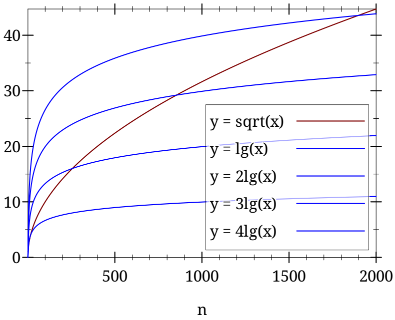

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\lg n} = \lim_{n\to\infty} \frac{n^{1/2}}{(\lg n)^1}$$

$$= \infty \qquad \text{(LRR2)}$$