Facet Generation and Symmetric Triangulation

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Facet enumeration up to symmetry

Definition

Linear transformation A is a *restricted automorphism* for $\text{cone}(V)$ if

$$\{ Av \mid v \in V \} = V$$

$\overline{\text{Aut}}(V)$ denotes the group of restricted automorphisms of $\text{cone}(V)$.

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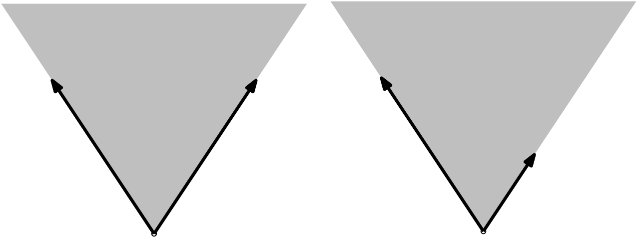

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Problem

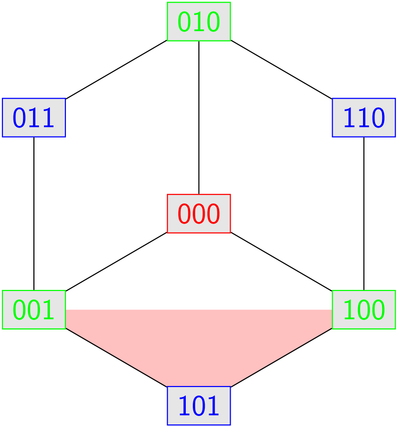
Given $V \subseteq \mathbb{R}^d$, $\overline{\operatorname{Aut}}(V)$.

Find One representative of each orbit of facet defining inequalities for $\operatorname{cone}(V)$.

Bases and Orbits

basis $(r-1)$ rays (d vertices) spanning a facet.

orbits $\overline{\text{Aut}}(P)$ acts on bases



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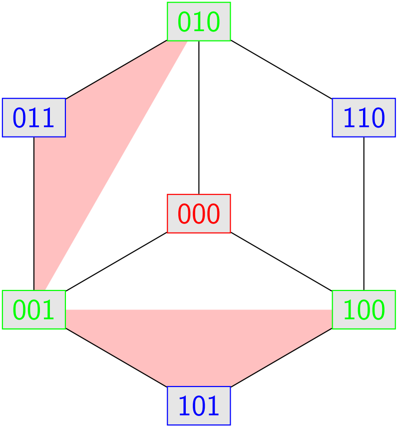

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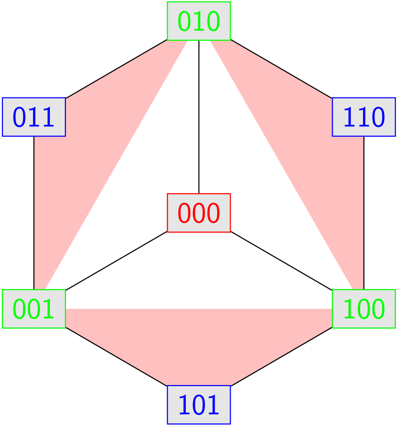
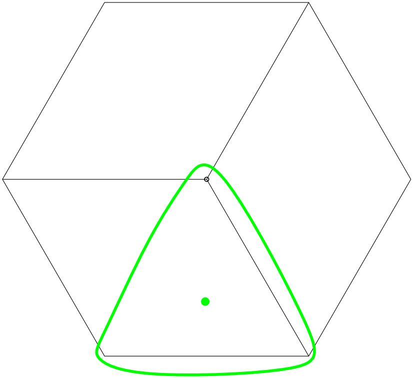


Exploring the Basis Graph

pivot $C' = C \setminus \{ l \} \cup \{ e \}$ such that C' is a basis.

basis graph nodes = bases, edges = pivots



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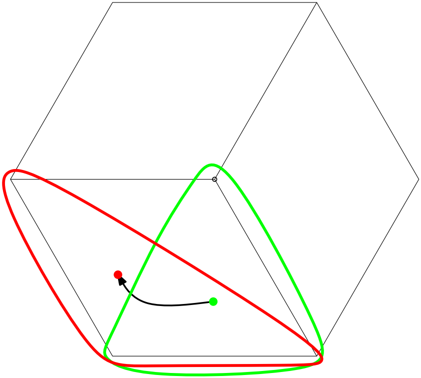

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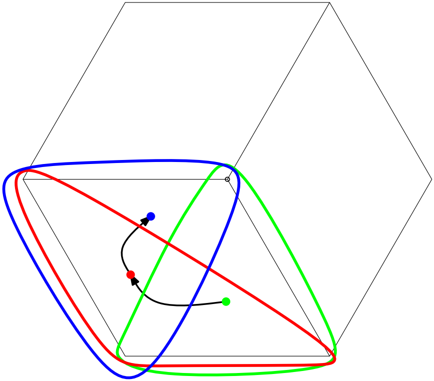

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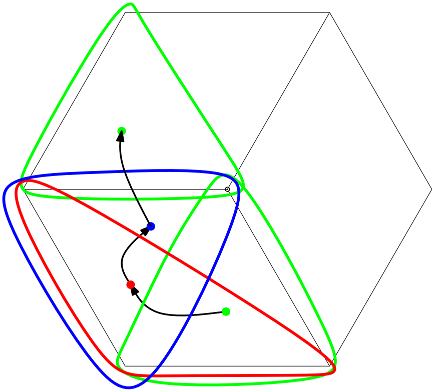

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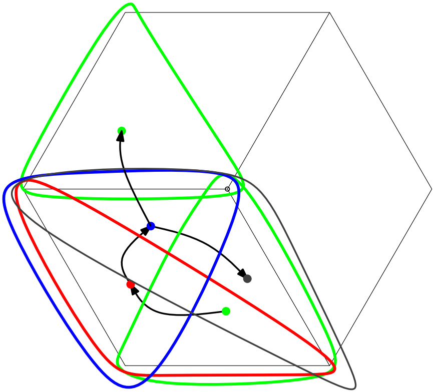

Wreath products

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Let $P = \text{conv}(v_1 \ldots v_m) \subset \mathbb{R}^d$. Let $Q = \text{conv}(w_1 \ldots w_n) \subset \mathbb{R}^e$.

$$P \wr Q = \text{conv} \begin{bmatrix} P & 0 & 0 & & 0 \\ 0 & P & 0 & & 0 \\ 0 & 0 & P & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & P \\ w_1 & w_2 & w_3 & & w_n \end{bmatrix}$$

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Roughly, $\overline{\mathrm{Aut}}(Q)$ acts *on* "big columns" and $\overline{\mathrm{Aut}}(P)$ *within* them.

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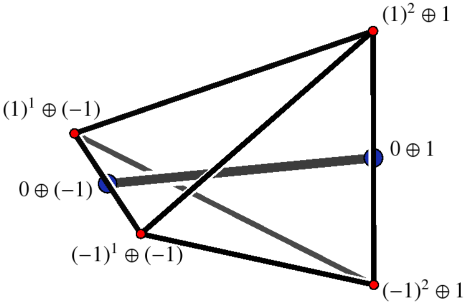

$[-1,1] \wr [-1,1]$
(courtesy of Joswig and Lutz (2005))

Wreath Products of Cross Polytopes

Example

Let $C_k = \operatorname{conv}\{\pm e_1, \ldots, \pm e_k\}$. Let $P = C_d \wr C_e$.

- P has dimension $D = 2de + e$ and $4de \sim 2D$ vertices
- P has $2^{(d+1)e}$ facets, each containing $3de \sim 1.5D$ vertices
- P has one orbit of vertices, facets, and $(D-1)$-bases.

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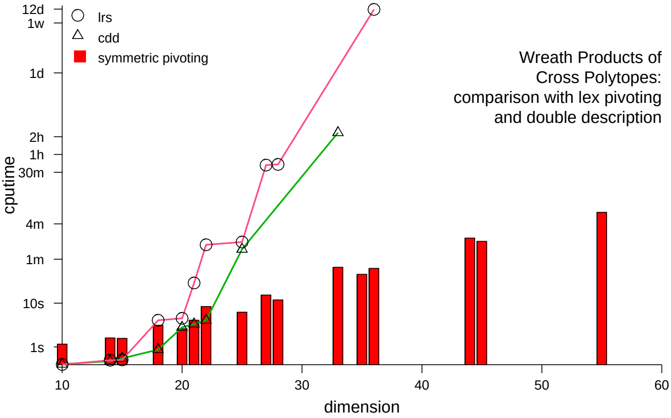


Orbitwise Degenerate Polytopes

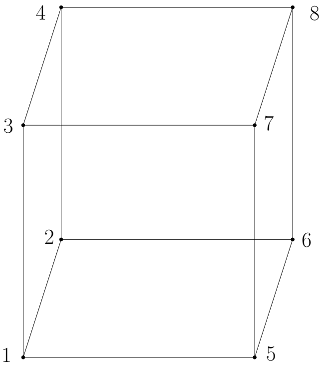
	Dimension	Triangulation $\triangle$s	Basis Orbits
Cut	10 ($n = 5$)	496	2
	15 ($n = 6$)	186636	6300
	4	48	4
	5	240	17
Cubes	6	1440	237
	7	10080	9892
	8	80640	> 209000

Valid Perturbation

Definition

$\widetilde{V}$ is a *valid perturbation* of V if $\exists \nu(\cdot) : V \leftrightarrow \widetilde{V}$ such that $\forall W \subseteq V$,

1. If $\nu(W)$ is linearly dependent then W is.
2. If $\nu(W)$ is extreme for $\widetilde{V}$ then W is extreme for V.



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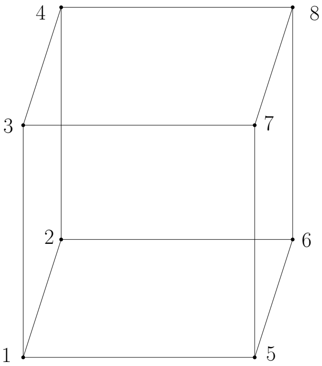

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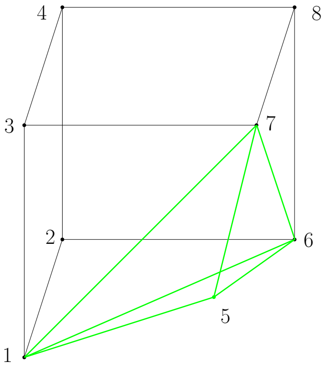

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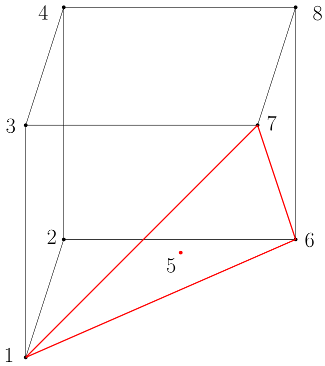


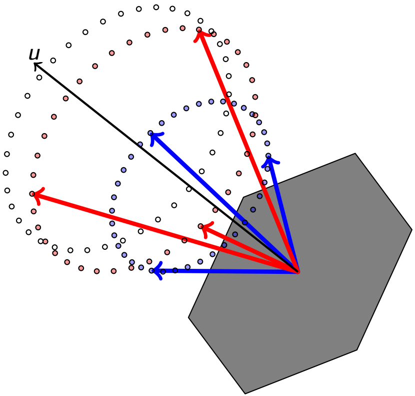
Symmetry preserving perturbation

Proposition

- *Let $V \subset \mathbb{R}^d$. $V_1, \ldots, V_k$ the orbits of V under H, and u be a fixed point for H,*
- *There exists $\varepsilon_1 \gg \cdots \gg \varepsilon_k \geq 0$ such that*

$$V' = \bigcup_j (V_j \pm \varepsilon_j u)$$

is a valid perturbation of V and $H \leq \overline{\mathrm{Aut}}(V')$.



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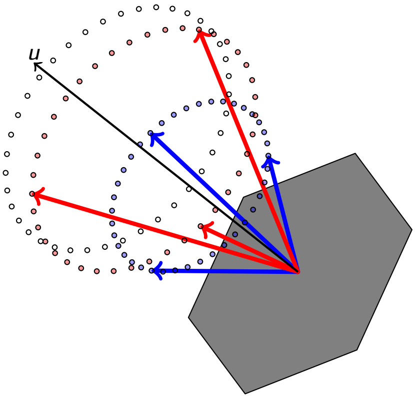

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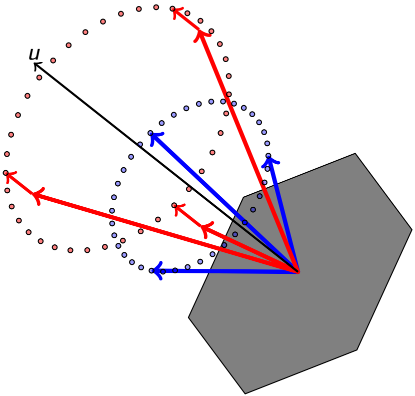
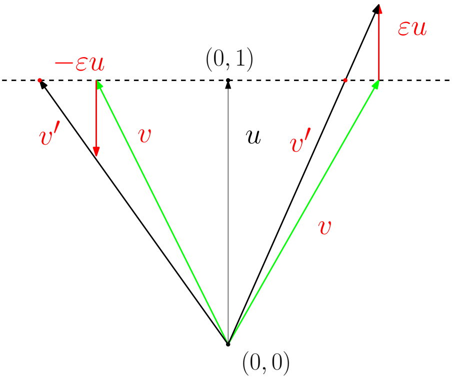


Affine Orbitwise Perturbation

Perturbation by Scaling

Linear	Affine	Name
$(p, 1) + \varepsilon(0, \ldots, 0, 1)$	$\dfrac{p}{1+\varepsilon}$	push
$(p, 1) - \varepsilon(0, \ldots, 0, 1)$	$\dfrac{p}{1-\varepsilon}$	pull



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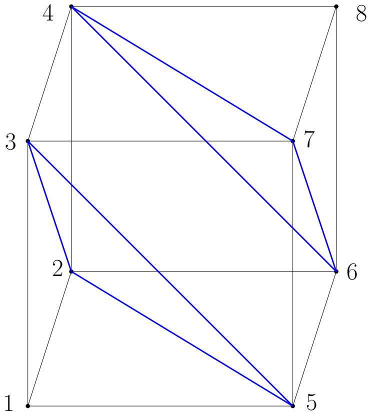

push(1, 8)

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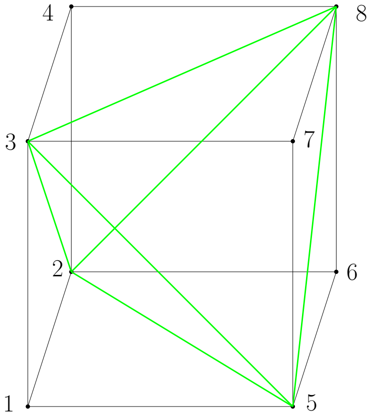


pull$(2, 3, 5, 8)$

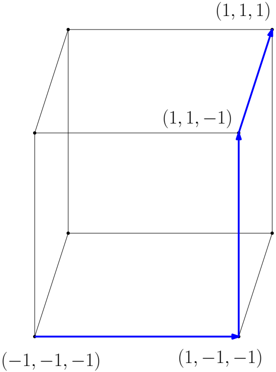
Linear Ordering Triangulation

Definition

- Let $I^d = [-1,1]^d$. Let $\mathbf{e} = (1,\ldots,1)$.
- For each $\rho \in \mathrm{Sym}(d)$, there is a path $[\rho]$ from $-\mathbf{e}$ to $\mathbf{e}$.
- Define Δ_ρ as $\mathrm{conv}[\rho]$.
- The *linear ordering triangulation* of bdy I^d is the intersection of bdy I^d with all Δ_ρ

Proposition

$H_d = \mathrm{stab}(\overline{\mathrm{Aut}}(I^d), \{-\mathbf{e}, \mathbf{e}\})$ *acts transitively on the l.o.t. of* bdy I^d.



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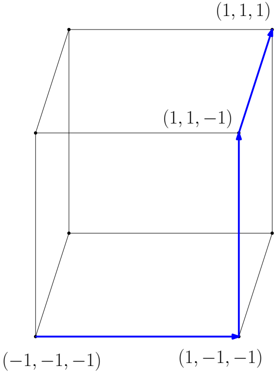

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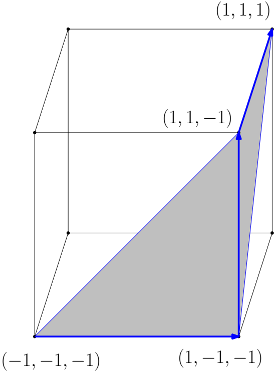

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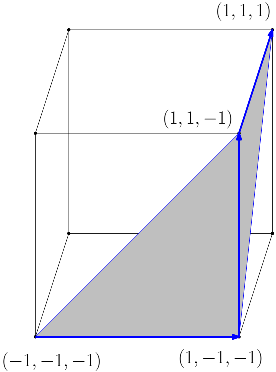

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Linear Ordering Perturbation

Example

Let $\tilde{I}^d$ denote the H_d-orbitwise pulling of I^d in order induced by $\omega(v) = \min(\mathbf{e}^T v, -\mathbf{e}^T v)$.

- bdy $\tilde{I}^d \equiv$ the l.o.t. of bdy I^d.
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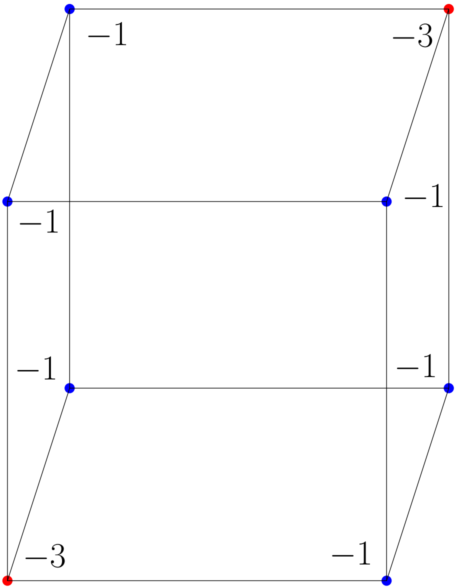

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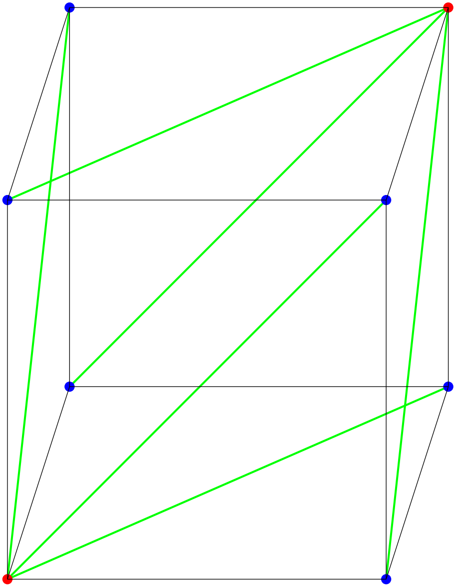

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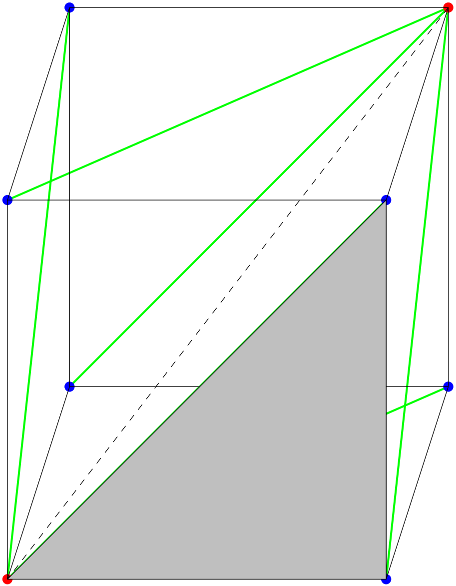

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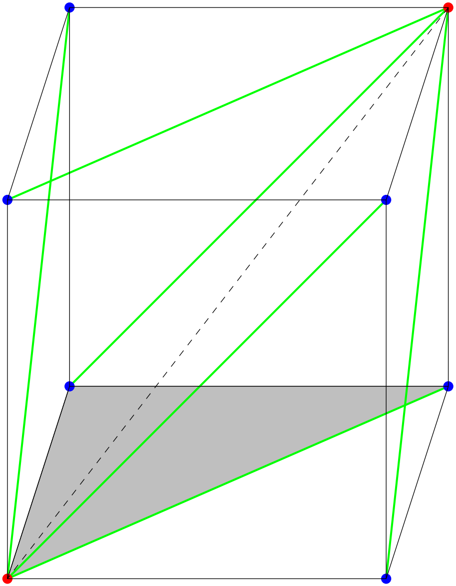

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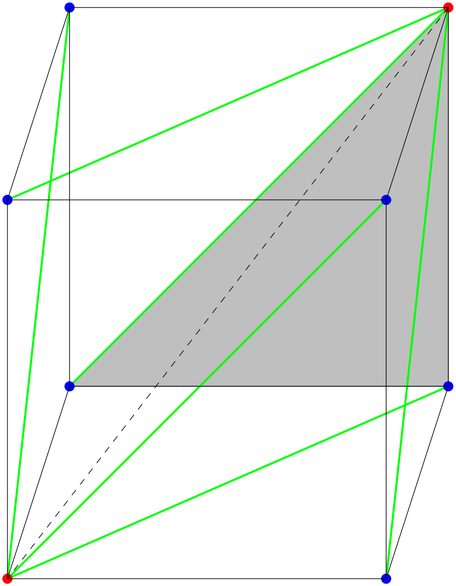
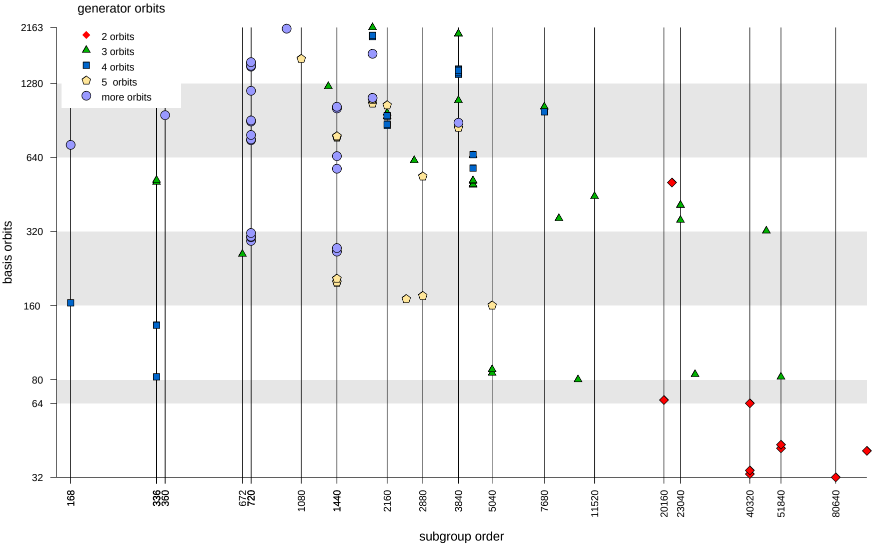


Example: E_7 root lattice contact polytope

Contact Polytope for E_7 root lattice

		Orbits
Dimension	8	
Group Order	2903040	
Vertices	126	1
Facets	632	2
lrs Δ's	20520	
bases		161

What makes a good subgroup?



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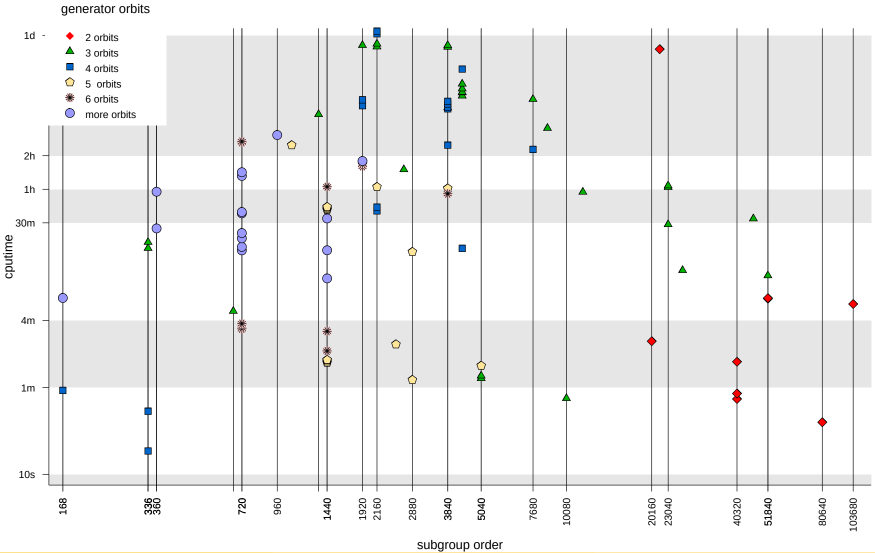

Conclusions

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