Succinct linear programs for easy problems

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Joint Work with D. Avis, H.R. Tiwary, and O. Watanabe

Outline

1. Matchings and Matching Polytopes
2. Extension Complexity
3. Weak extended formulations
4. From circuits to LPs
5. Connections with non-negative rank
6. Constructing an LP from Pseudocode

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Edmonds' matching polytope EM_n

Matchings

Given graph $G = (V, E)$, $M \subseteq E$ is a *matching* if every $v \in V$ is contained in at most one $e \in M$. M is *perfect* if $|M| = |V|/2$.

Edmonds' Matching Polytope

$\mathrm{EM}_n = \mathrm{conv}\{\,\chi(M) \in \{0,1\}^{\binom{n}{2}} \mid M$ matching in $K_n\}$. Linear description consists of degree bounds, and for every $W \subset V$, $|W| = 2k + 1$, $k \geq 1$,

$$\sum_{e \subset W} x_e \leq (|W| - 1)/2$$

Edmonds' perfect matching polytope EP_n

Edmonds' perfect matching polytope EP_n

- $\mathrm{EP}_n = \mathrm{EM}_n \cap \{x \mid \mathbb{1}^T x = n/2\}$
- Linear description has degree bounds and for every $W \subset V$, $|W| = 2k+1$, $k \geq 1$,

$$\sum_{i \in W, j \notin W} x_{ij} \geq 1$$

- Face of EM_n by definition.
- Not hard to see equivalance of two odd set constraints for binary values.

Another perfect matching polytope

$$\psi(x) = \begin{cases} 1 & x \text{ char. vec. of graph with a perfect matching} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{PM}_n = \mathrm{conv}\{(x, \psi(x)) : x \in \{0,1\}^{\binom{n}{2}}\}$$

Proposition

EP_n is a face of PM_n and can be defined by

$$\mathrm{EP}_n = \{x : (x, w) \in \mathrm{PM}_n \cap \{\mathbb{1}^T x + (1 - w)n^2 = \frac{n}{2}\}\}$$

- The minimal graphs containing perfect matchings are the matchings themselves.

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Extensions and extended formulations

Definition

Polyhedron $Q \in \mathbb{R}^{d+e}$ is an *extension* of $P \subset \mathbb{R}^d$ if $P = TQ$ for some linear transform T

Definition

An *extended formulation* (EF) of a polytope $P \subseteq \mathbb{R}^d$ is a linear system

$$Ex + Fy = g, \;\; y \geqslant \mathbf{0} \tag{1}$$

such that $P = \{x \mid \exists y \; Ex + Fy = g\}$

- In both cases the *size* is the number of inequalities / facets.
- All but a small number of equations can be eliminated (in some sense non-constructive).

Slack matrices

Suppose

$$
\begin{aligned}
P_{\text{in}} &\subseteq P_{\text{out}} \subseteq \mathbb{R}^n \\
P_{\text{in}} &= \operatorname{conv}(\{v_1, \ldots, v_k\}) \\
P_{\text{out}} &= \{x \in \mathbb{R}^n \mid a_i^\top x \leqslant b_i, 1 \leqslant i \leqslant m\}
\end{aligned}
$$

then

$$
\begin{aligned}
S_{ij}(P_{\text{out}}, P_{\text{in}}) &= b_i - a_i^\top v_j \\
S(P) &= S(P, P)
\end{aligned}
$$

Of course these matrices are generally huge!

Nonnegative rank

Definition

The nonnegative rank $\mathrm{rank}_+(S)$ of a matrix S is the smallest r such that there exist $T \in \mathbb{R}_+^{f \times r}, U \in \mathbb{R}_+^{r \times v}$ and $S = TU$

Theorem (Y91)

The following are equivalent

- *$S(P)$ has non-negative rank at most r*
- *P has extension size at most r*
- *P has an EF of size at most r.*

Symmetric Extended Formulations

- Extended formulation $Q(x, y)$ is *symmetric* if every permutation π of the coordinates of x extends to a permutation of y that preserves Q.

Theorem (Yanakakis91)

The matching polytope has no polynomial size symmetric extended formulation.

No extended formulation of EP_n is succinct

Theorem (Rothvoß2013)

Any extended formulation of the perfect matching polytope EP_n has complexity $2^{\Omega(n)}$.

- This takes as a starting point the idea of covering the support of the slack matrix with rectangles of 1s (Y91).
- Slack matrices are also useful in proving the following, which shows that EP_n and PM_n also have exponential extension complexity.

Lemma (FMPTW2012)

Let P, Q and F be polytopes. Then the following hold:

- *if F is an extension of P, then $\mathrm{xc}(F) \geq xc(P)$*
- *if F is a face of Q, then $xc(Q) \geq xc(F)$.*

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Optimizing over PM_n

For a given input graph $G(\bar{x}) = (V, E)$ we define $c = (c_{ij})$ by:

$$c_{ij} = 1 \quad ij \in E \qquad c_{ij} = -1 \quad ij \notin E \qquad 1 \leqslant i < j \leqslant n$$

Let d be a constant such that $0 < d \leqslant 1/2$.

$$z^* = max\ z = c^T x + dw \tag{2}$$
$$(x, w) \in \mathrm{PM}_n$$

Proposition

For $\bar{x} \in \{0,1\}^{\binom{n}{2}}$, the optimum solution to (2) is unique, and

$$z^* = \begin{cases} \mathbb{1}^T \bar{x} + d & \text{if } G(\bar{x}) \text{ has a perfect matching} \\ \mathbb{1}^T \bar{x} & \text{otherwise} \end{cases}$$

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Polytopes for decision problems

Consider a decision problem defined by its *characteristic function*

$$\psi(x) = \begin{cases} 1 & x \text{ char. vec. of } \textbf{YES} \text{ instance} \\ 0 & \text{otherwise} \end{cases}$$

For each input size q we can define a polytope

$$P(\psi, q) = \mathrm{conv}\{(x, \psi(x)) : x \in \{0,1\}^q\}$$

To optimize, we will use $c^T x + dw$, d small constant, $c = \phi(\bar{x})$

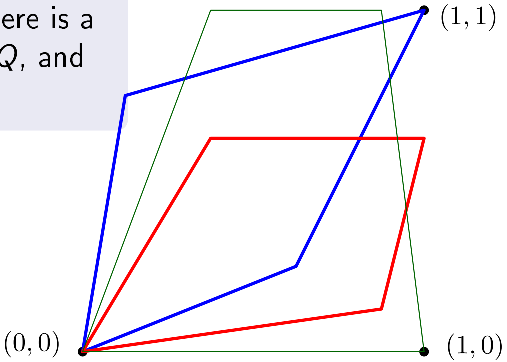
$$\phi(x)_i = \begin{cases} 1 & \text{if } x_i = 1 \\ -1 & \text{if } x_i = 0 \end{cases}$$

0/1-property

Definition

Let $Q \subseteq [0,1]^{q+t}$ be a polytope. We say that Q has the *x-0/1 property* if

- For each x in $\{0,1\}^q$ there is a unique vertex (x, y) of Q, and
- $(x, y) \in \{0,1\}^{q+t}$.



Weak Extended Formulation

Definition

Define polytope Q by $x \in [0,1]^q$, $w \in [0,1]$, $s \in [0,1]^r$ and

$$Ax + bw + Cs \leqslant h$$

For any $\bar{x} \in \{0,1\}^q$, $0 < d \leq 1/2$

$$z^* = \max \ \{\phi(\bar{x})^T x + dw : (x, w, s) \in Q\} \qquad (3)$$

Let $m = \mathbb{1}^T \bar{x}$. Q is a *weak extended formulation (WEF)* of $P(\psi, q)$ if Q has the x-0/1 property, and

- For every *YES* instance the solution to (3) is unique and $z^* = m + d$.
- For every *NO* instance $z^* < m + d$ and for all sufficiently small d, $z^* = m$ and is the solution to (3) is unique.

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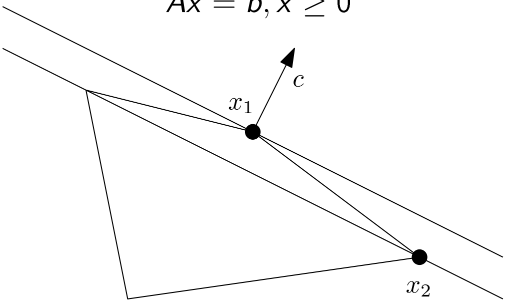
Objective function gaps

$$\max c^T x$$
$$Ax = b, x \geq 0$$

Let $A \in \mathbb{Z}^{m \times n}, c \in \mathbb{Z}^n, b \in \mathbb{Z}^m$ define an LP.

- Let B_1 and B_2 be basis matrices
- $x_i = B_i^{-1} b$
- Cramer's rule, integrality, $x_1 \neq x_2$



$$\Delta = c^T(x_1 - x_2) \geq \frac{1}{|B_1||B_2|}$$

- Let $\sigma = \max_{i,j} \mathrm{abs}(a_{ij})$
- Using the Hadamard bound, $|B_i| \leq \sigma^m m^{m/2}$,

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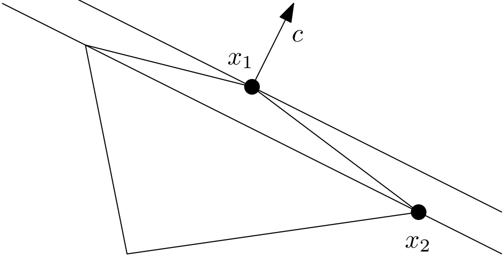

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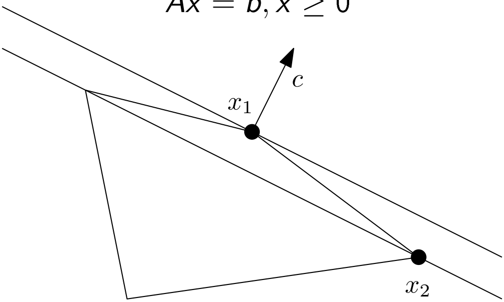

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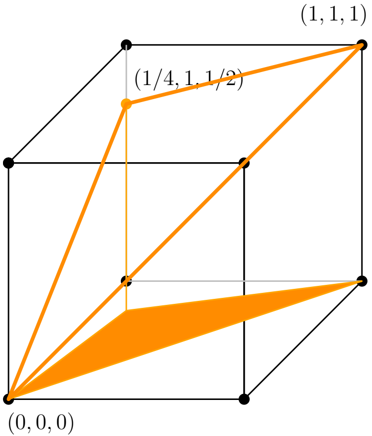
A simple example

$$\mathrm{PM}_2 = \mathrm{conv}\{(0,0),(1,1)\}$$
$$Q_2 = \mathrm{conv}\{(0,0,0),(1,1,1),(1/4,1,1/2)\}$$

$d = 1/2$

- $\bar{x} = 1$: $c_{12} = 1$ and $z = c^T x + dw$ gets same on P_2 and Q_2
- $\bar{x} = 0$: $z^* = 0 = m$ over PM_2 and $z^* = 1/4 < 1/2 = m + d$ over Q_2



$0 < d < 1/4$

$\bar{x} = 0$: $z^* = 0 = m$ over both PM_2 and Q_2

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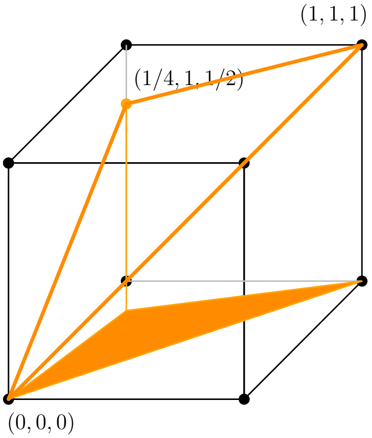

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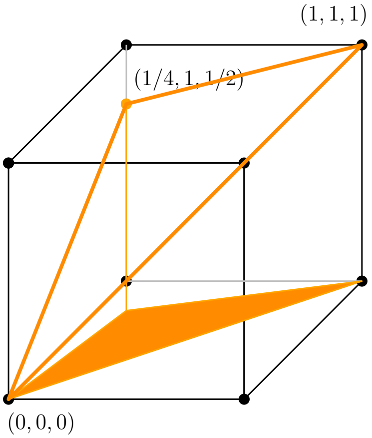

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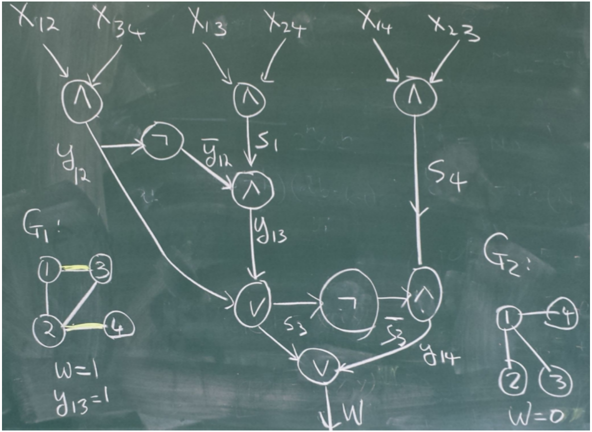
Circuits

Definition

A *(boolean) circuit* with q input bits $x = (x_1, x_2, \dots, x_q)$ is defined by a sequence of *gates* $x_i = x_j \circ x_k$ $(\circ \in \{\vee, \wedge\})$ or $x_i = \neg x_j$ where $i > j, k, q$.



P/Poly and P

Definition

P/Poly is the class of decision problems with polynomial sized circuits for each input size.

Definition

A family C_n of circuits is *polynomial-time uniform* if there exists a deterministic Turing machine M that on input 1^n generates C_n in polynomial time.

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Valiant's Construction I

$x_i = x_j \wedge x_k,$

$$\begin{aligned} x_j + x_k - x_i &\leqslant 1 \\ -x_j \qquad + x_i &\leqslant 0 \\ - x_k + x_i &\leqslant 0 \\ x_i &\geqslant 0 \end{aligned} \qquad \text{(AND)}$$

$x_i = x_j \vee x_k$

$$\begin{aligned} -x_j - x_k + x_i &\leqslant 0 \\ x_j \qquad - x_i &\leqslant 0 \\ x_k - x_i &\leqslant 0 \\ x_i &\leqslant 1 \end{aligned} \qquad \text{(OR)}$$

Valiant's Construction II

- Given circuit C of size t, let polytope $Q(C)$ be constructed using systems (AND) and (OR), and by substituting $x_i = \neg x_j$ by $1 - x_j$.
- $Q(C)$ has $4t$ inequalities and $q + t$ variables, and all coefficients $0, \pm 1$

Lemma (Valiant1982)

Let C be a boolean circuit with q input bits $x = (x_1, x_2, ..., x_q)$ and t gate output bits $y = (y_1, y_2, ..., y_t)$. $Q(C)$ has the x-0/1 property and for every input x the value computed by C corresponds to the value of y_t in the unique extension $(x, y) \in Q$ of x.

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WEFs from Circuits

Lemma

Let ψ be a decision problem. Let C_n be a (not necessarily uniform) family of circuits for ψ. $Q(C_n)$ is a weak extended formulation for $P(\psi, n)$.

Proposition

Every decision problem in **P/poly** *admits a weak extended formulation Q of polynomial size.*

In principle, a matching polytope

- Perfect Matching is in **P**, therefore we can construct one circuit C_n per input size.
- From the circuit, we can construct $Q(C_n)$ which is a WEF for PM_n; no poly size extension exists

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Polytopal sandwiches for languages

Switch to terminology of *language*

$$L = \{x \in \{0,1\}^* \mid \psi(x) = 1\}$$
$$L(n) = \{x \in \{0,1\}^n \mid \psi(x) = 1\}$$

For $L \subseteq \{0,1\}^*$ define a pair of *characteristic functions*

$$\psi(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases},$$
$$\phi(x)_i = \begin{cases} 1 & \text{if } x_i = 1 \\ -1 & \text{if } x_i = 0 \end{cases}$$

And a pair of polytopes

$$V(L(n)) = \operatorname{conv}(\{(x, \psi(x)) \mid x \in \{0,1\}^n\}.$$
$$H(L(n)) := \left\{ (x, w) \;\middle|\; \begin{array}{ll} \phi(a)^\intercal x + dw \leqslant a^\intercal \mathbb{1} + d & \forall a \in L(n) \\ \phi(a)^\intercal x + dw \leqslant a^\intercal \mathbb{1} & \forall a \notin L(n) \end{array} \right\}$$

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Slack matrices for languages

$$M_{a,b}(L(n)) = a^\intercal \mathbb{1}_n - 2a^\intercal b + \mathbb{1}^\intercal b + \alpha(a,b)$$

where

$$\alpha(a,b) = \begin{cases} d & \text{if } a \in L, b \notin L \\ -d & \text{if } a \notin L, b \in L \\ 0 & \text{otherwise} \end{cases},$$

$$M(L(n)) = S(H(L(n)), V(L(n))) \tag{4}$$

Concise coordinates

Concise coordinates

- A matrix or vector X has *concise coordinates* with respect to n (X *is* $\mathrm{cc}(n)$) if each element has a binary encoding bounded by a polynomial in n.
- A polytope is $\mathrm{cc}(n)$ if its vertex and facet matrices are.

Extended formulations with concise coordinates

- $\mathrm{rank}_+^n(M)$ denotes the minimum rank of a non-negative factorization $M = ST$ such that S and T are both $\mathrm{cc}(n)$.
- $\mathrm{xc}^n(P)$ denotes the minimum number of inequalities in a $\mathrm{cc}(n)$ extended formulation of P.

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Optimizing over sandwiches

Lemma

Let P be a polytope such that $V(L(n)) \subseteq P \subseteq H(L(n))$. Then, deciding whether a vector $a \in \{0,1\}^n$ is in L or not can be achieved by optimizing over P along the direction $(\phi(a), d)$ for some constant $0 < d \leqslant 1/2$.

Main idea

Objective values for P are sandwiched between those for $V(L(n))$ and $H(L(n))$.

Rank of Sandwiches

Lemma

Let $P_{\text{in}} = \text{conv}(V)$, $P_{\text{out}} = \{x \mid Ax \leq b\}$ *be* $\text{cc}(n)$.

$$\text{rank}^n_+(S) = \min\{\, \text{xc}^n(P) \mid P \text{ is } \text{cc}(n) \text{ and } P_{\text{in}} \subseteq P \subseteq P_{\text{out}} \,\}$$

Proof sketch.

$$S' := \begin{bmatrix} S(P) & S(P, P_{\text{out}}) \\ S(P_{\text{in}}, P) & S(P_{\text{in}}, P_{\text{out}}) \end{bmatrix} \qquad (\leq)$$

$$\text{rank}^n_+(S(P_{\text{in}}, P_{\text{out}})) \leq \text{rank}^n_+(S') = \text{rank}^n_+(S(P)) = \text{xc}^n(P)$$

$$S(P_{\text{in}}, P_{\text{out}}) = TU$$

$$Q = \{(x, y) \mid Ax + Ty = b, y \geq \mathbf{0}\} \qquad (\geq)$$

$$P = \{x \mid \exists (x, y) \in Q\}$$

$$P_{\text{in}} \subseteq P \subseteq P_{\text{out}}$$

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$$\mathrm{rank}_+^n(S(P_{\mathrm{in}}, P_{\mathrm{out}})) \le \mathrm{rank}_+^n(S') = \mathrm{rank}_+^n(S(P)) = \mathrm{xc}^n(P)$$

$$\begin{aligned} S(P_{\mathrm{in}}, P_{\mathrm{out}}) &= TU \\ Q &= \{(x, y) \mid Ax + Ty = b, y \ge \mathbf{0}\} \qquad (\ge) \\ P &= \{x \mid \exists (x, y) \in Q\} \\ P_{\mathrm{in}} \subseteq P &\subseteq P_{\mathrm{out}} \end{aligned}$$

Rank of Sandwiches and **P/Poly**

Theorem

$L \in \mathbf{P/Poly}$ *iff* $\mathrm{rank}_+^n(M(L(n)))$ *is polynomial in* n

(only if).

- Small circuits C_n implies WEF $Q(C_n)$. Define $P = \mathrm{proj}_{(x,w)}(Q(C_n))$
- $V(L(n)) \subseteq P \subseteq H(L(n))$
- $\mathrm{rank}_+^n(M(L(n))) = \mathrm{rank}_+^n(S(H(L(n)), V(L(n)))) \leq \#\mathrm{ineq}(Q)$

□

(if).

- Small rank_+^n implies $\exists P \; V(L(n)) \subset P \subset H(L(n))$
- P has small extension Q, optimize over Q to decide L.

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Outline

1. Matchings and Matching Polytopes
2. Extension Complexity
3. Weak extended formulations
4. From circuits to LPs
5. Connections with non-negative rank
6. Constructing an LP from Pseudocode

Constructing Concrete Polytopes

- We want to construct actual polytopes for the perfect matching problem and other problems in P.
- Algorithms are typically expressed as pseudocode and not circuits; directly designing a circuit for Edmonds' algorithm seems nontrivial.
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Key ideas for compiling pseudocode to LPs

- Based on binary variables, with integrality guarantees propagated by induction, as in the circuit case.
- A *step counter* is modelled as a set of boolean variables, which enable and disable the constraints modelling each line of code.
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